Section One: Calculator-free **35% (53 Marks)**

This section has **eight (8)** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time for this section is 50 minutes.

Question 1 **(6 marks)**

The polynomial $h(z) = z^4 - 4z^3 + 17z^2 - 16z + a$, where a is a real constant, is exactly divisible by $(z - 2i)$.

(a) Determine the value of a. (2 marks)

$h(2i) = 16 + 32i - 68 - 32i + a$ ✓

$= -52 + a = 0$

$\therefore a = 52$ ✓

or

	1	-4	17	-16	a
$z = 2i$	1	$-4+2i$	$-8i+13$	$26i$	0

$\Rightarrow -52 + a = 0$

(b) Write down two zeros (roots) of $h(z)$. (1 mark)

$\pm 2i$ ✓

(c) Determine the other zeros of $h(z)$. (3 marks)

$h(z) = (z^2+4)(z^2 + az + b)$ ✓

$= (z^2+4)(z^2 - 4z + 13)$ ✓

or

	1	$-4+2i$	✓ $-8i+13$	$26i$
$-2i$	1	-4	✓ 13	0

$\Rightarrow$ other zeros are $\dfrac{4 \pm \sqrt{16-52}}{2}$

$= 2 \pm \dfrac{\sqrt{-36}}{2}$

$= 2 \pm 3i$ ✓

Question 2 **(5 marks)**

Let $v = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$.

(a) Express v in polar form. (2 marks)

$v = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i$

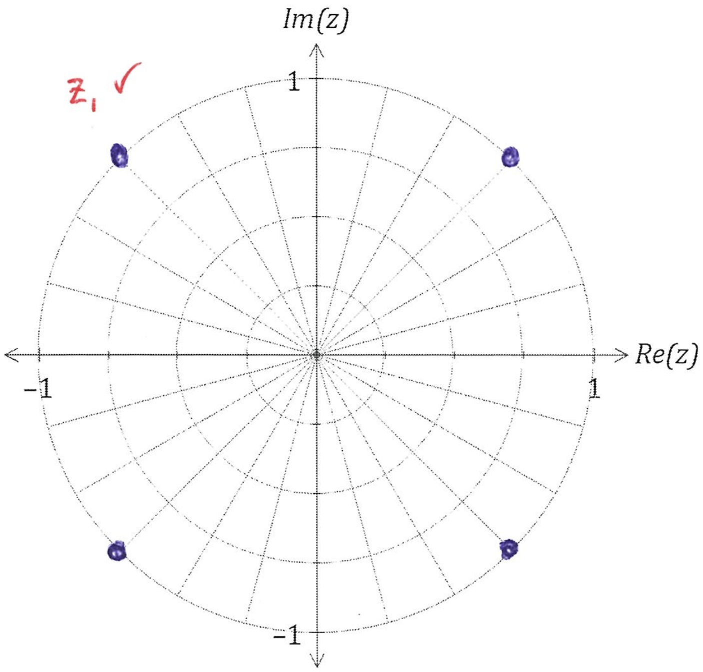
$= cis\left(-\frac{\pi}{4}\right)$ or $\left(1, -\frac{\pi}{4}\right)$ ✓ ✓

(b) Show that $v^4 = -1$. (1 mark)

$v^4 = cis(-\pi)$ ✓

$= -1$

(c) Plot the roots of $z^4 + 1 = 0$ on the following Argand diagram. (2 marks)



✓ $z_2 \rightarrow z_4$

Question 3 **(8 marks)**

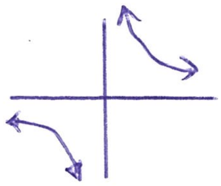
Two functions are defined by $f(x) = \sqrt{3x - 1}$ and $g(x) = \frac{1}{x}$.

(a) Determine the composite function $f(g(x))$ and the domain over which it is defined. (3 marks)

$f \circ g(x) = \sqrt{\frac{3}{x} - 1}$ ✓

Domain $x \neq 0$ and $\frac{3}{x} - 1 \geq 0$

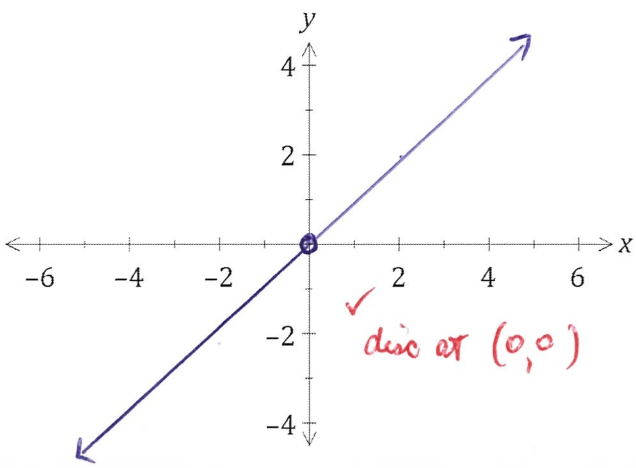
$\frac{3}{x} \geq 1$



$\Rightarrow \quad 0 < x \leq 3$

✓ ✓

(b) Sketch the graph of $y = g(g(x))$ on the axes below. (2 marks)



$y = \frac{1}{\frac{1}{x}} = x$ ✓ , $x \neq 0$

(c) ~~(d)~~ The graph of $y = f(x) = \sqrt{3x - 1}$ is shown below, passing through point P with coordinates $(2.62, 2.62)$.
Determine an equation for $f^{-1}(x)$, the inverse of $f(x)$, and sketch the graph of $y = f^{-1}(x)$ on the same axes. (3 marks)

$y = \sqrt{3x - 1}$

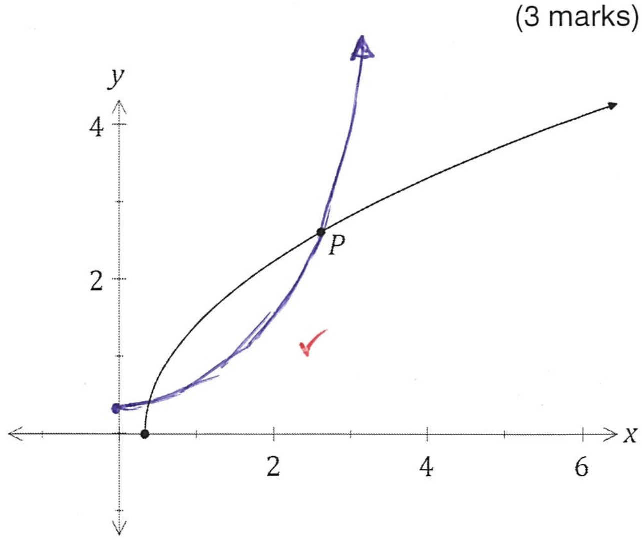
has inverse

$x = \sqrt{3y - 1}$ ✓

$x^2 = 3y - 1$

$y = \frac{x^2 + 1}{3}$

ie $f^{-1}(x) = \frac{x^2 + 1}{3}$ ✓ , $x \geq 0$



Question 4 **(8 marks)**

The function f is defined as $f(x) = \frac{x^2-1}{x^2+1}$.

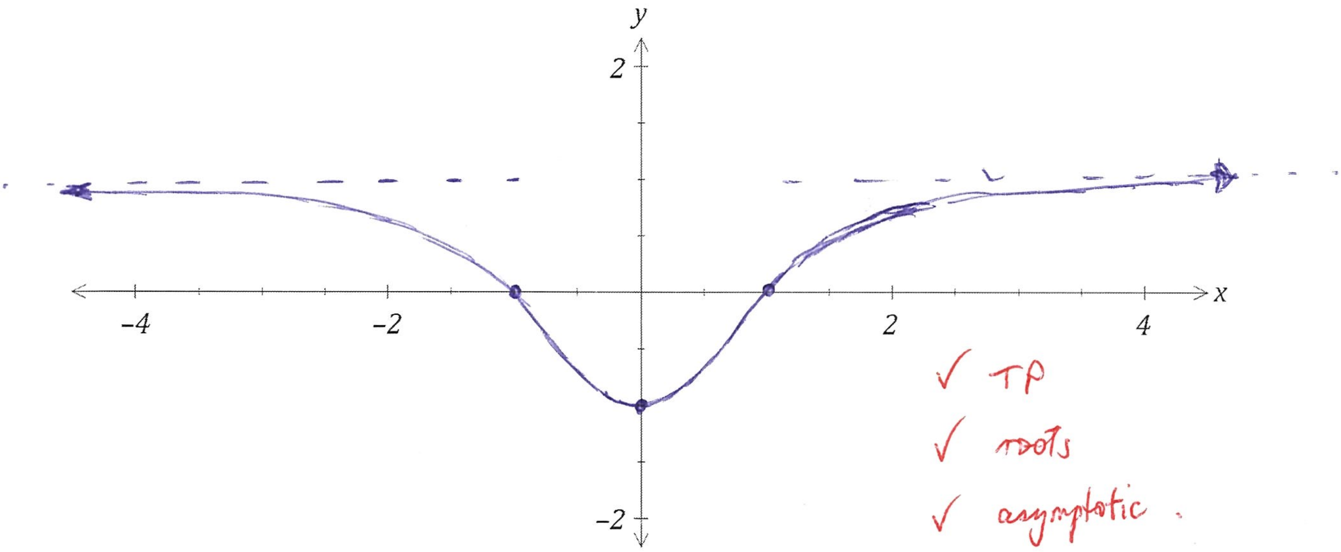
(a) Show that the **only** stationary point of the function occurs when $x = 0$. (2 marks)

$$f'(x) = 0 \Rightarrow \frac{2x(x^2+1) - 2x(x^2-1)}{(x^2+1)^2} = 0 \quad ✓$$

$$\Rightarrow 4x = 0 \quad ✓ \quad \text{since } (x^2+1)^2 > 0$$

$$\Rightarrow x = 0 \text{ is unique}$$

(b) Sketch the graph of $y = f(x)$ on the axes below. (3 marks)



(c) Using your graph, or otherwise, determine all solutions to

(i) $f(x) = |f(x)|$. (1 mark)

$x \geq 1$ or $x \leq -1$ ✓

(ii) $f(x) = f(|x|)$. (1 mark)

$x \in \mathbb{R}$ True for all x. ✓

(iii) $f(x) = \frac{1}{f(x)}$. (1 mark)

$x = 0$ ✓

Question 5 **(7 marks)**

(a) Using partial fractions, or otherwise, determine $\int \frac{x-19}{(x+1)(x-4)}\,dx$. (4 marks)

$$\frac{x-19}{(x+1)(x-4)} = \frac{a}{x+1} + \frac{b}{x-4}$$

$$a + b = 1$$
$$-4a + b = -19$$
$$5a = 20$$
$$a = 4$$
$$b = -3$$

$$\therefore \int \frac{4}{x+1} - \frac{3}{x-4}\,dx = 4\ln|x+1| - 3\ln|x-4| + c$$

$$= \ln\frac{|x+1|^4}{|x-4|^3} + c$$

(b) Use the substitution $u = \sin x$ to evaluate $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{\cos x}{\sqrt{\sin x}}\,dx$. (3 marks)

$$= \int_{\frac{1}{2}}^{1} \frac{du}{\sqrt{u}}$$

$u = \sin x$

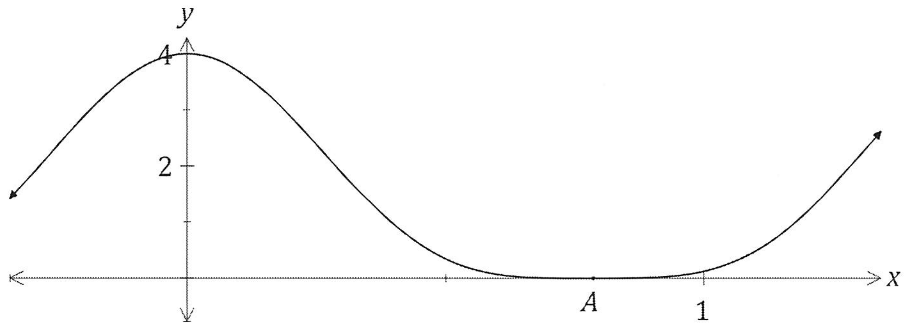
$du = \cos x\,dx$

$$= 2\sqrt{u}\,\Big|_{\frac{1}{2}}^{1}$$

$$= 2 - \frac{2}{\sqrt{2}} \quad \text{or} \quad 2 - \sqrt{2}$$

Question 6 **(7 marks)**

The graph of $y = f(x)$ is shown below, where $f(x) = 4\cos^4(2x)$ and A is the smallest root of $f(x),\ x > 0$.



(a) Show that $4\cos^4(2x) = \frac{3+4\cos(4x)+\cos(8x)}{2}$. (3 marks)

$$LHS = 4\left(\tfrac{1}{2}\cos 4x + \tfrac{1}{2}\right)^2 \quad \text{since} \quad \cos^2 2x = \tfrac{1}{2}\cos 4x + \tfrac{1}{2}$$

$$= \cos^2 4x + 2\cos 4x + 1$$

$$= \tfrac{1}{2}\cos 8x + \tfrac{1}{2} + 2\cos 4x + 1$$

$$= \frac{3 + 4\cos 4x + \cos 8x}{2} \quad \text{as required}$$

(b) Hence determine $\int 4\cos^4(2x)\,dx$. (2 marks)

$$= \frac{3x}{2} + \frac{\sin 4x}{2} + \frac{\sin 8x}{16} + C$$

(c) Use the formula $V_x = \pi\int_a^b y^2 dx$ to write a definite integral to represent the volume of the solid generated when the region bounded by $y = f(x)$, $y = 0$, $x = 0$ and $x = A$ is rotated through 360° about the x-axis. (2 marks)

$$V_x = \pi \int_0^{\frac{\pi}{4}} 16\cos^8 2x \, dx$$

$4\cos 2x = 0$

$\cos 2x = 0$

$2x = \frac{\pi}{2}$

Question 7 **(7 marks)**

(a) Show that the gradient of the curve $2x^2 + y^2 = 3xy$ at the point $(1, 2)$ is 2. (3 marks)

$$4x + 2y\frac{dy}{dx} = 3y + 3x\frac{dy}{dx} \checkmark \checkmark$$

Subst $(1,2)$ $\quad 4 + 4\frac{dy}{dx} = 6 + 3\frac{dy}{dx}$

$$\therefore \frac{dy}{dx} = 2 \checkmark$$

i.e. gradient at $(1,2)$ is 2

(b) Another curve passing through the point $(-2, 10)$ has gradient given by $\frac{dy}{dx} = \frac{2xy}{1+x^2}$. Use a method involving separation of variables and integration to determine the equation of the curve. (4 marks)

$$\int \frac{dy}{y} = \int \frac{2x}{1+x^2}\,dx \checkmark$$

$$\ln y = \ln|1+x^2| + c \checkmark$$

$$y = c_1(1+x^2) \checkmark$$

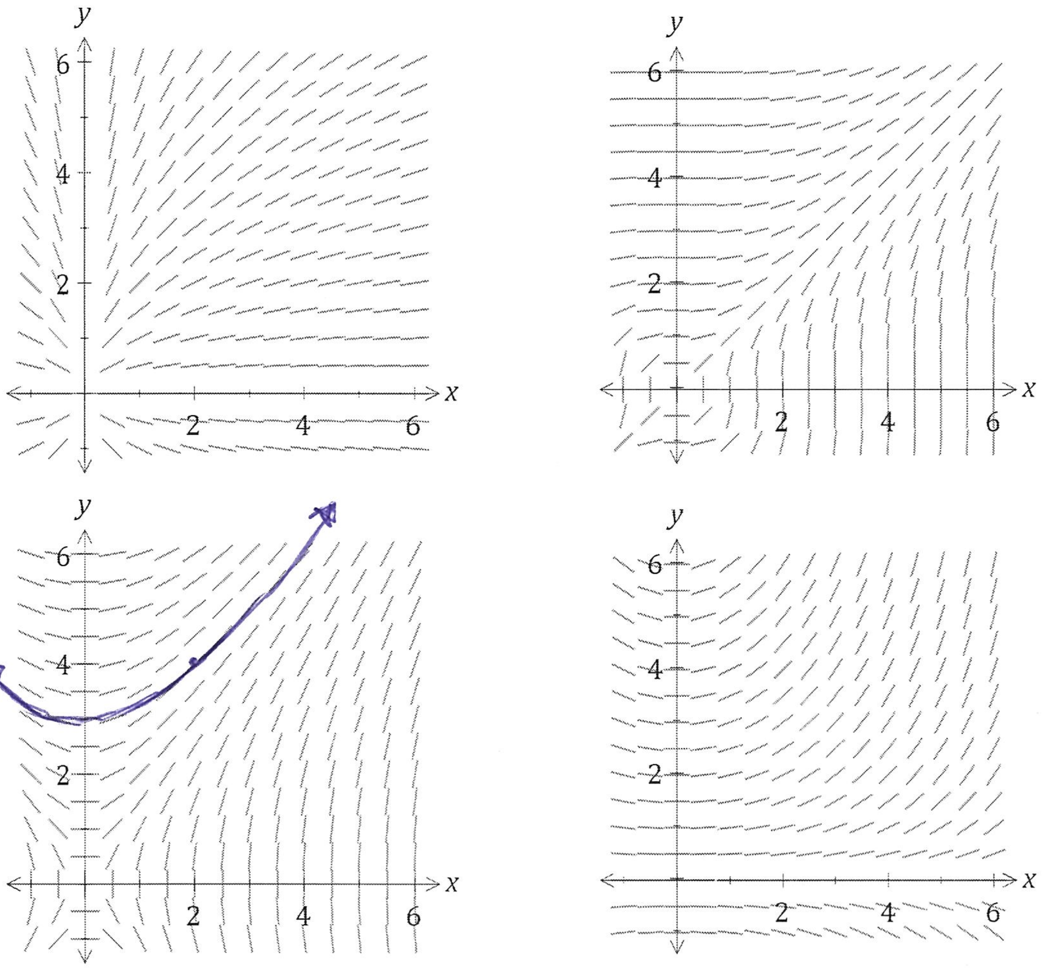
$(-2,10) \Rightarrow \quad 10 = c_1(1+4)$

$$c_1 = 2$$

$$\therefore y = 2 + 2x^2 \checkmark$$

Question 8 **(5 marks)**

The differential equation $y' = \frac{2x}{y}$ is shown in just one of the four slope fields below.



(a) On the slope field for $y' = \frac{2x}{y}$, sketch the solution of the differential equation that passes through the point $(2, 4)$. (3 marks)

Graph 3, as shown. ✓

(2,4) ✓

shape ✓

(b) Another solution to the differential equation passes through the point $(6, -3)$. Use the incremental formula $\delta y \approx \frac{dy}{dx} \times \delta x$, with $\delta x = \frac{1}{10}$, to estimate the y-coordinate of this curve when $x = 6.1$. (2 marks)

$y = -3 + \frac{2x}{y} \times 0.1$ ✓ at $(6, -3)$

$= -3 - 4 \times 0.1$

$= -3.4$ ✓

i.e. at $(6.1, -3.4)$

End of questions